AT THE INTERSECTION OF DIVERSIFICATION, VOLATILITY AND CORRELATION

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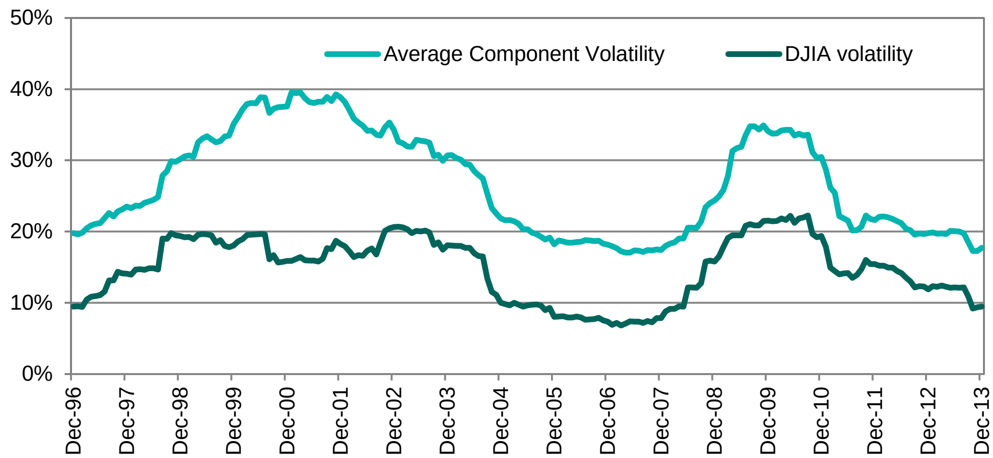
What is the "benefit" of diversification? It is certainly not an improvement in returns: if you think that the average stock will return 10% this year, you probably wouldn't expect a return much better (or worse) than 10% from a diversified portfolio of stocks. If you "know" that a particular stock will outperform, you may believe that you would be better off concentrating your position: diversification is a distraction from the main event.[1]

But for an investor without an opinion as to which of two similar securities might perform better, the option of holding both may appear appealing: the average return, with less risk than either. **The benefit of diversification is ultimately a question of risk reduction.** Such "free lunch" aspects of diversified portfolios are so well known that they are often overlooked. Yet their rapid reduction in the highly correlated markets of the last five years suggests that **whatever free lunch we've been getting, it seems to be smaller than it used to be**. This suspicion is well grounded, as we'll illustrate via several real-world examples that follow.

The major goal of this paper is to investigate **what drives the diversification benefit.** Our work on **dispersion**[2] inspires the approach here; as documented below, we shall see that this simple single-period measure, with some modest mathematical manipulation, parallels the diversification benefit.

Exhibit 1(a): Dow Jones Industrial Average Index™ and Component Volatility (1996 - 2013)



Source: S&P Dow Jones Indices. Index and index-weighted 24-month volatility (annualized) shown for the period Dec. 31, 1996 to Dec. 31, 2013. Charts are provided for illustrative purposes. Past performance is no guarantee of future results.

[1] To readers who "know" that a particular stock will outperform, we can only echo the Psalmist (139:6): "Such knowledge is too wonderful for me; It is high, I cannot attain unto it."

[2] Edwards, Tim and Craig J. Lazzara, "Dispersion: Measuring Market Opportunity," December 2013.

RESEARCH

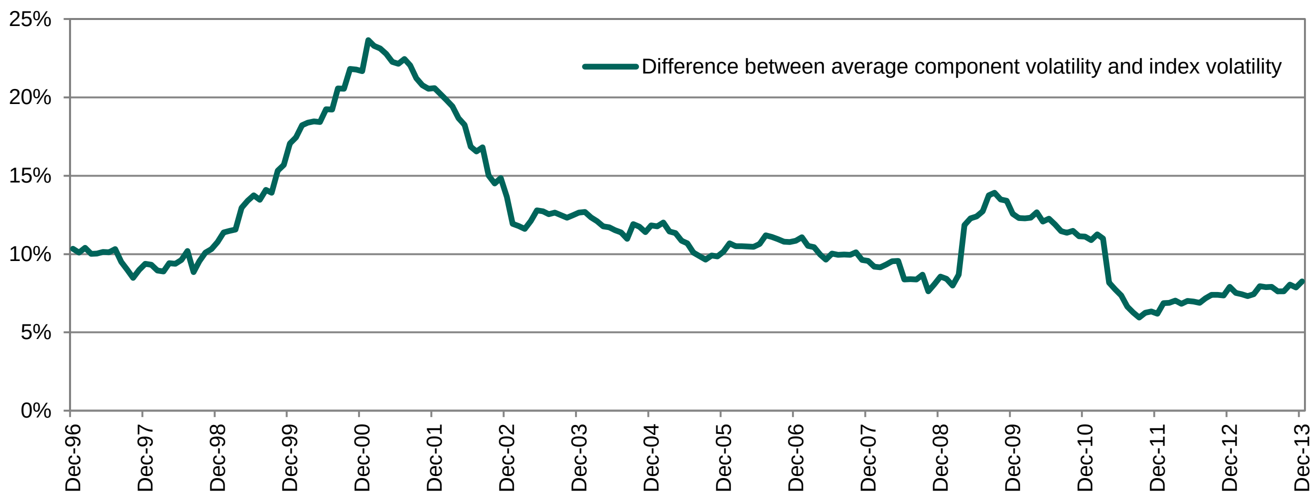
Experience—common sense, even—suggests that diversification must be related to correlation, and perhaps also to market volatility. But correlation and volatility are measures of history; their measurement requires a time series of data. Dispersion, as we've previously considered it, is a single-period measure. Averaging dispersion's impact over time proves to be the final ingredient in the central equations showing an **analytic relationship among diversification, volatility and correlation.**

In Section 1, we'll define the diversification benefit and relate it to the dispersion among index constituents. Specifically, we'll show how dispersion is related to the residual risks that—together with market volatility—complete our understanding of the total risk in individual securities. In Section 2, we'll relate correlation to the balance between dispersion and volatility. Finally, we'll conclude with some remarks on these relationships in the current market environment and its potential evolution.

SECTION 1: QUANTIFYING DIVERSIFICATION

The precise benefit of diversification is easily illustrated by example. Over the two years ending in December 2013, the least volatile stock in the Dow Jones Industrial Average (DJIA) was McDonald's Corporation, with a volatility of 11%.[3] **But the volatility of the DJIA as a whole was only 9%**—less, in other words, than the volatility of its least volatile component. If the diversification benefit is understood as solely a question of risk, we can quantify it as the difference (reduction) in volatility between a portfolio's constituents and their combination. In later sections, we will more commonly use variance, but Exhibits 1(a) and 1(b) retain the concept of volatility. The upper, lighter line in Exhibit 1(a) is the average DJIA component volatility. Note that to compare apples to apples (here and throughout) we use index-weighted averages. The equivalent volatility for the DJIA itself (in Exhibit 1a) is the lower, darker line. Exhibit 1b shows this difference explicitly.

Exhibit 1(b): Dow Jones Industrial Average Index "Diversification Benefit" (1996 - 2013)



Source: S&P Dow Jones Indices. Data shows the difference between the two series provided in Exhibit 1(a). Charts are provided for illustrative purposes. Past performance is no guarantee of future results.

The "diversification benefit"—the gap between the two lines of Exhibit 1(a) shown in Exhibit 1(b)—has been relatively low for some time, at least for the DJIA. And explicitly plotting the historical range provides precedent; on a historical basis such lows look unlikely to change dramatically in the short run and yet unlikely to persist indefinitely. Greater precision is certainly possible: decomposing stock returns into a "market return" plus some individual "alpha" will be the first step in accounting for such variations.

Return, Alpha and Diversifiable Risks

Much of the economic theory about the stock market conveniently assumes that the risks of investing in a particular stock can be split into two categories: those risks specific to the company in question—the quality of

[3] Measured with two years of monthly data (2012 – 2013).

management, mergers, new product launches and so on—plus a second component, aggregating the overall macroeconomic conditions facing companies in general. It is reasonable to hope that such an attribution can identify precisely those risks that are diversifiable. Of course, on a forward-looking basis, it is difficult—perhaps impossible—to disentangle these risks completely. Will that new product launch be successful in an economic downturn? Will a booming stock market camouflage management's mistakes?

From a backward-looking standpoint, attribution is simpler. We can—if it suits us—simply define each stock's return over each time period as a sum:

$$R_{i,t} = M_t + \alpha_{i,t}$$

Here, and throughout, $R_{i,t}$ denotes the return of stock *i* over the period *t* (assuming a multi-period history is the object of study) while M_t represents the "market" return for each period. The second term $\alpha_{i,t}$ represents the idiosyncratic "alpha" that is the relative performance of that stock versus the market. Conventionally, one also assumes that the market itself is represented by some combination of its components—a fixed[4] system of weightings w_i so that:

$$M_t = \sum_i w_i R_{i,t}$$

Defining v^2 as the variance of the market portfolio over the full time period, and weighting averages by market weights, some manipulation of these two equations[5] identifies the average variance of the stocks in the market as a combination of v^2 and the average variance of the alphas:

EQUATION 1 - THE DIVERSIFICATION EQUATION:

$$\boldsymbol{Average\ component\ variance} = v^2 + \sum_i w_i Var(\alpha_{i,t})$$

We have interpreted "the diversification benefit" as the difference in volatility between a portfolio and its components. If we are content to deal with variances as opposed to their square root, volatility, **Equation 1 relates the difference in the two variances to that of the "alphas."**

Of course, simply subtracting the variance of the market from that of the average stock—and hence divining the diversification benefit without the aid of Equation 1—is not a Herculean task. While it's true that Equation 1 does not meaningfully shorten the calculation, it is nonetheless of interest for a particular reason. While the estimation of volatility (at either stock or index level) requires a *time series* of performance data, the attribution of the diversification benefit derived using Equation 1 can be *approximated from a single time period* via dispersion.

Before we justify the preceding statement, it is worth stressing its importance. One of dispersion's key advantages is that its computation does not require a long time series of data. We can calculate an index's dispersion for last month (or for yesterday) based only on last month's (or yesterday's) constituent returns and weights. In contrast, computing the same index's volatility, or the average volatility of the index's constituents, requires a time series of returns: five years' worth of monthly returns or one month of daily returns, etc. The longer the data series, the more robust the estimate will be—and the longer it will take for any changes to be reflected. Otherwise said, using five years of monthly returns may give us a reliable estimate of monthly volatility. But if something changes next month, it will take a long time for us to notice it. **Changes in dispersion will show up much sooner.** [6]

[4] Weightings are not usually fixed and instead vary over time (e.g., with market capitalization). The approximation is valid for most indices over reasonably short time periods; the simplification is adopted here as it provides considerable mathematical convenience.

[5] See Appendix A.

[6] For an early treatment of this idea, as well as the later relationship to correlations, see Solnick, Bruno and Jacques Roulet, "Dispersion as Cross-Sectional Correlation," *Financial Analysts Journal*, January/February 2000, pp. 54-61.

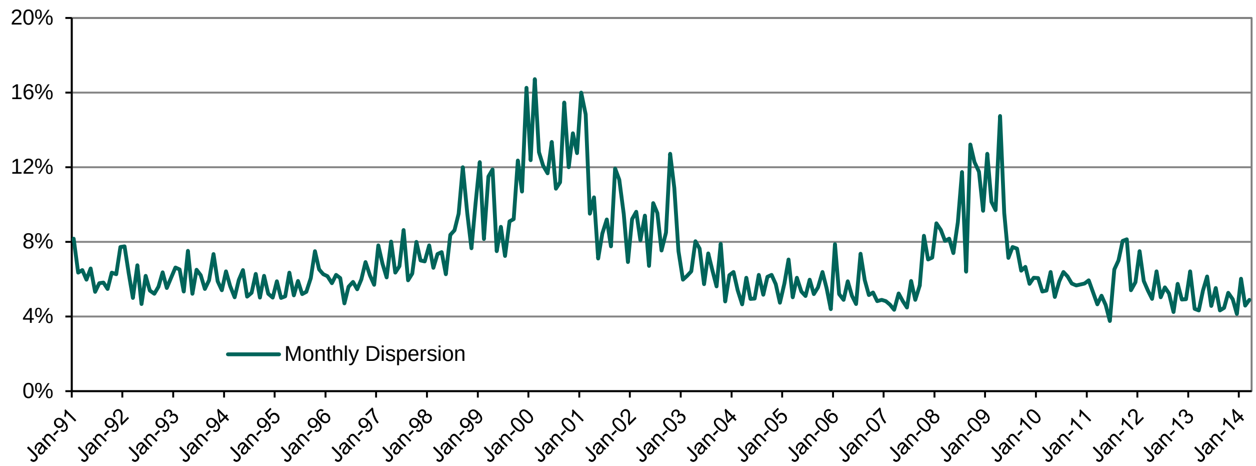
Short-Term Measures of the Diversification Benefit

During a single time period, each stock records a single return and a single alpha, and thus in itself tells us very little about volatility. However, looking *cross-sectionally* (across all stocks) provides a broad yet contemporary sample of how much variation exists during a given time interval. Dispersion, which measures this cross-sectional variation, is calculated over a single time period (using the notations already introduced) as:

$$Dispersion \quad = \sqrt{\sum_{i=1}^{n} w_i \left(R_{i,t} - M_t\right)^2} \, .$$

The behavior of monthly dispersion for the S&P 500® is illustrated in Exhibit 2.

Exhibit 2: S&P 500 Monthly Dispersion (Jan. 1991 - Mar. 2014)



Source: S&P Dow Jones Indices. Charts are provided for illustrative purposes. Past performance is no guarantee of future results.

Some intuition may be helpful in interpreting dispersion. For each stock, $\left(R_{i,t} - M_t\right)$ identifies the degree of movement in that stock that was not reflected in the market overall. The dispersion calculation—aggregating and averaging all such cancelled-out movements—is therefore immediately suggestive of a diversification benefit. **Dispersion measures the amount of variation that is *lost* when considering the market portfolio as a single investment**, as opposed to a collection of separate components. Moreover, the form of the equation defining dispersion will remind at least some readers of the definition of standard deviation, which tells us that what dispersion measures may well relate to volatility. Bringing together these comments, our intuition (and hope) is that a single-period measure of the variation among alphas may be reasonably expected to provide clues as to the potential volatility reduction in the market portfolio **over time**.

More concretely, since we want to investigate the interaction of dispersion and volatility, we need to match the (cross-sectional, single-period) computation of dispersion with the (time series) computation of volatility. We can accomplish this by averaging our periodic dispersion calculations over the same period that we use to compute volatility. We'll call that the *average dispersional variance,* denote it by d^2 and define it as:

$$d^2 = \ Average\ over\ all\ periods\ of\ (Dispersion^2)$$

This definition, coupled with some algebra[7], lets us substitute (d^2) for the rightmost term in Equation 1 to derive:

EQUATION 2: THE ROLE OF DISPERSION IN DIVERSIFICATION:

$$\boldsymbol{Average\ component\ variance = v^2 + d^2}$$

[7] See Appendix B. Unfortunately such equality is not generally exact—stocks with biased alpha cause an overstatement in d^2 (their *real* variance over time is not as much as might be inferred from their differences to the market). Equality in Equation 2 requires each stock to have a long-term average alpha of zero.

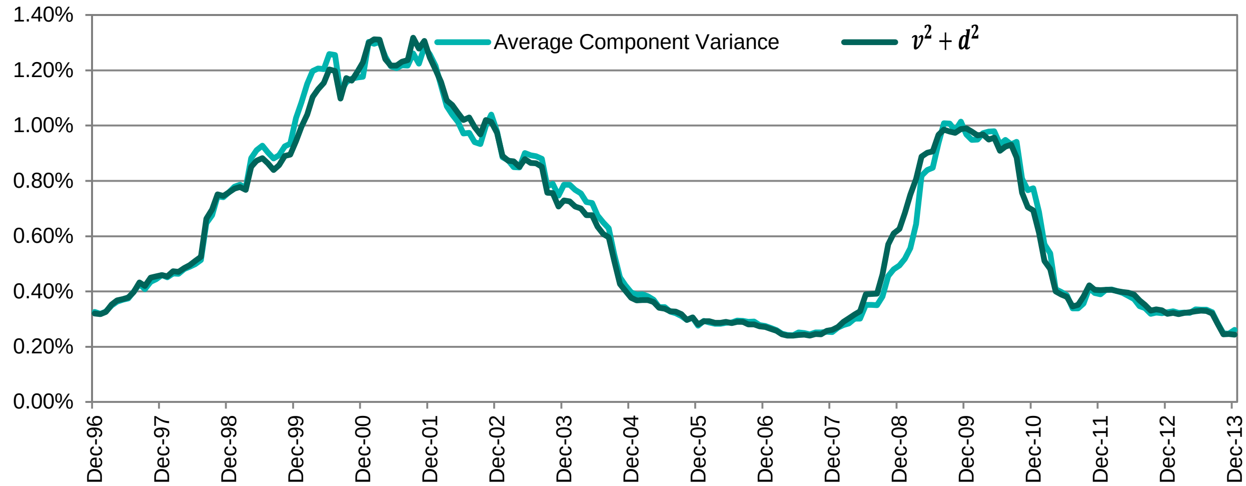
That is to say: **over time, dispersion measures the diversification benefit of the market portfolio**.

In order to derive a rigorously exact formula, we have incorporated several assumptions along the way. If we are to have confidence that the formula is at least *approximately* true in the real world, we need to test it in practice. Exhibits 3(a) and 3(b), demonstrate the accuracy of Equation 2 for our original example of the DJIA and for the more diversified S&P 500 on a more granular basis. In both cases, the numbers suggest an encouraging margin of error.

Empirical Evidence for Equation 2

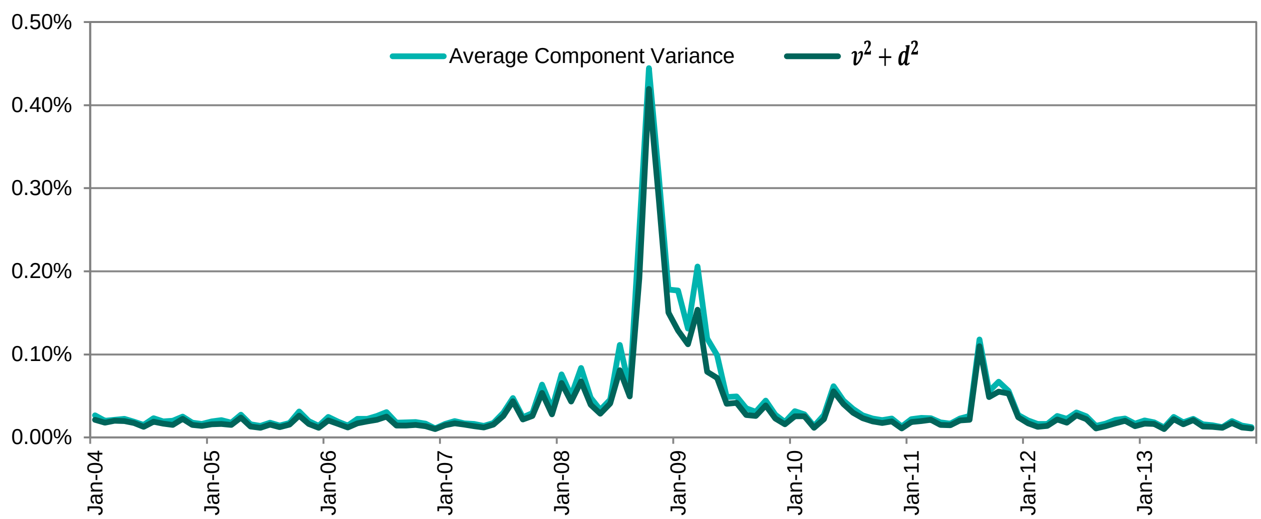
In order to avoid any potential confusion, note that Exhibit 1(b) shows *volatility,* while Exhibit 3(a) shows average single stock *variance* (the darker, green line) and compares it to the sum of (i) index variance and (ii) the time-average of dispersion squared during the period. The timescale of measurement for Exhibit 3(a) is a *rolling two years of monthly return* measurements, while in Exhibit 3(b) the timescale is shortened to a rolling one calendar month of daily return measurements (and hence the more precipitous changes).

Exhibit 3(a): Two Years/Monthly Dow Jones Industrial Average Statistics (Dec. 1996 – Dec. 2013)



Source: S&P Dow Jones Indices. Charts are provided for illustrative purposes. Past performance is no guarantee of future results.

Exhibit 3(b): Monthly/Daily S&P 500 Statistics (Dec. 2003 – Dec. 2013)



Source: S&P Dow Jones Indices. Charts are provided for illustrative purposes. Past performance is no guarantee of future results.

SECTION 2: THE ROLE OF CORRELATION

In this section we relate our previous results to correlation, traditionally considered the critical measure of diversified portfolios. Although we've been somewhat dismissive of correlation as an input to investment decisions elsewhere,[8] we have no intention of underplaying its role here. In fact, **correlation is the key concept that unlocks the interplay between and evolution of dispersion and volatility**.

Unlike volatility and dispersion, which are measures of magnitude, correlation is a ratio: a number between positive one and negative one. At the simplest level, correlation measures the degree of relationship between two assets, that is to say *how much of their movement is shared.* For an index of many assets, however, the computation becomes more complex—what we really want to measure is the average correlation of every index component with every other index component, weighted by their importance in the index. For a 500-stock index, this requires computing 124,750 separate pairwise correlations, and then calculating the weighted average of those.[9] Only with the advent of modern computing power did this sort of analysis become routine.

Conceptually, however, the notion of an index's average pairwise correlation is not so daunting. Simply stated, each asset's movement can be split into a shared component—measured by correlation—and a separate component, independent of the first and unique to the asset in question. In terms of our earlier decomposition of returns, the shared movement is represented by the portfolio (or market) return over the period being examined. We represented this in Equation 2:

$$Average\ component\ variance = v^2 + d^2$$

Here v^2 represents the variance of the market, or the variance of the *shared* movement of an index's components. And d^2 represents the average dispersion of the market's components, or the variance of the *unshared* movement in returns. Intuitively, then, **we can think of correlation as the proportion of total variance that is shared**, as formulaically expressed in Equation 3:

EQUATION 3: CORRELATION, VOLATILITY AND DISPERSION:

$$Average\ pairwise\ correlation \approx \left(\frac{v^2}{v^2+d^2}\right)$$

Here the symbol "≈" means "is approximately equal to." We deal with the irksomely squiggly nature of the equality sign by making certain (fairly common) assumptions; the interested reader will find the derivation in Appendix C. As with our earlier considerations, such model-based "proofs" are secondary to the question of how well the formulas actually work for real world markets.

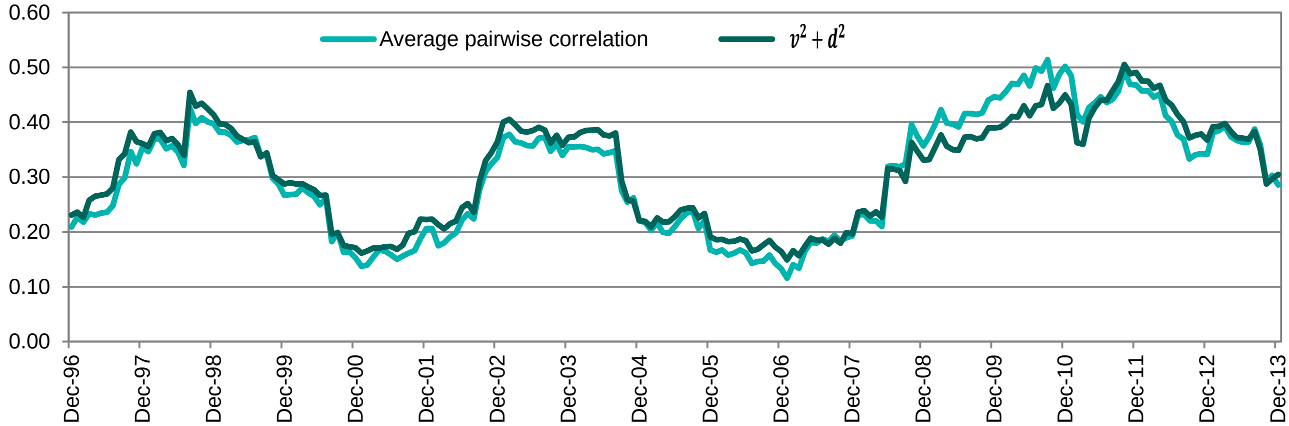
Empirical Evidence for Equation 3

As with our evidence for Equation 2, we test the accuracy of Equation 3 first with regard to the DJIA and at the granularity of monthly returns over a two-year rolling period; this is shown in Exhibit 4(a). In Exhibit 4(b), we test the equation's accuracy for the S&P 500 and at the granularity of daily returns over a calendar month.

[8] See Edwards and Lazzara, *op. cit.*, pp 7-8.

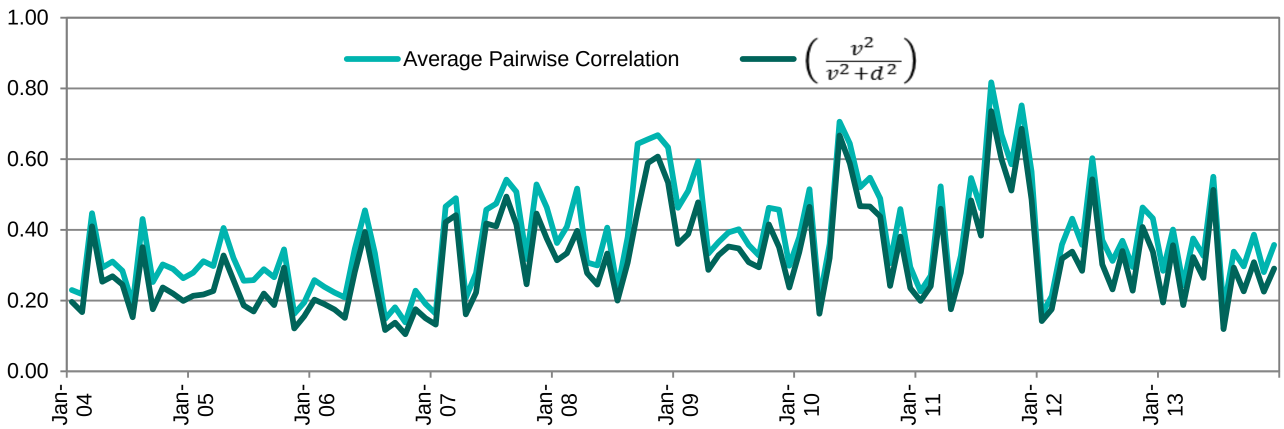
[9] For an N-stock index, there are N * (N-1)/2 pairwise correlations.

Exhibit 4(a): Two Years/Monthly Dow Jones Industrial Average Statistics (Dec. 1996 – Dec. 2013)



Source: S&P Dow Jones Indices. Charts are provided for illustrative purposes. Past performance is no guarantee of future results.

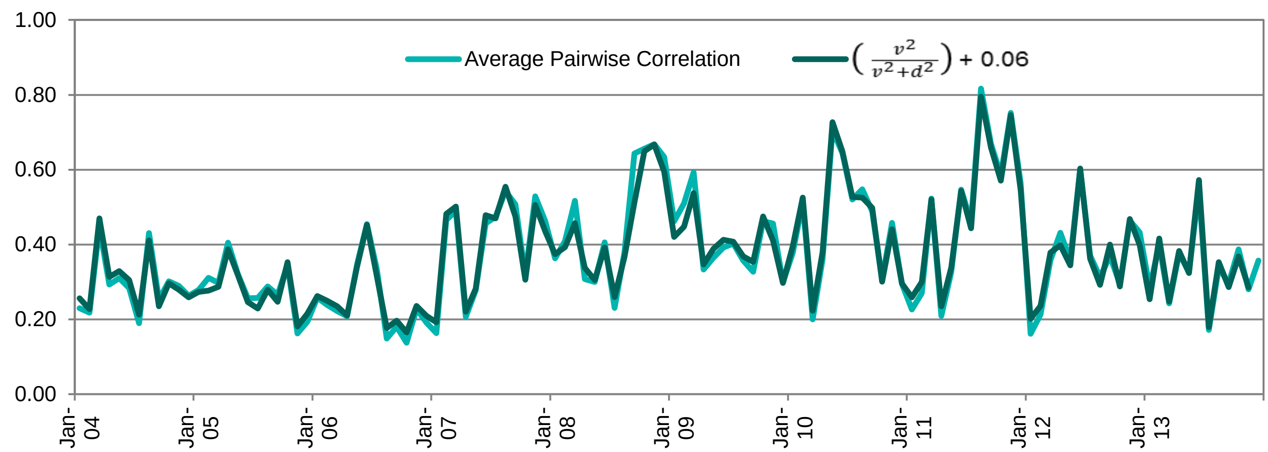
Exhibit 4(b): Monthly/Daily S&P 500 Statistics (Jan. 2004 – Dec. 2013)



Source: S&P Dow Jones Indices. Charts are provided for illustrative purposes. Past performance is no guarantee of future results.

We conclude our empirical evidence with the observation that Exhibit 4(b) demonstrates a somewhat consistent underestimation of average pairwise correlations; such consistency means that this inaccuracy might be fairly easily managed. Exhibit 4(c) shows the same data as Exhibit 4(b) with an adjustment—specifically adding the average underestimation of 0.06—and demonstrates the remarkably accuracy of this adjusted estimation.

Exhibit 4(c): Monthly/Daily S&P 500 Correlation Statistics (Jan. 2004 – Dec. 2013)



Source: S&P Dow Jones Indices. Charts are provided for illustrative purposes. Past performance is no guarantee of future results.

SECTION 3: CONCLUSIONS AND AN INTERPRETATION OF CURRENT MARKET DYNAMICS

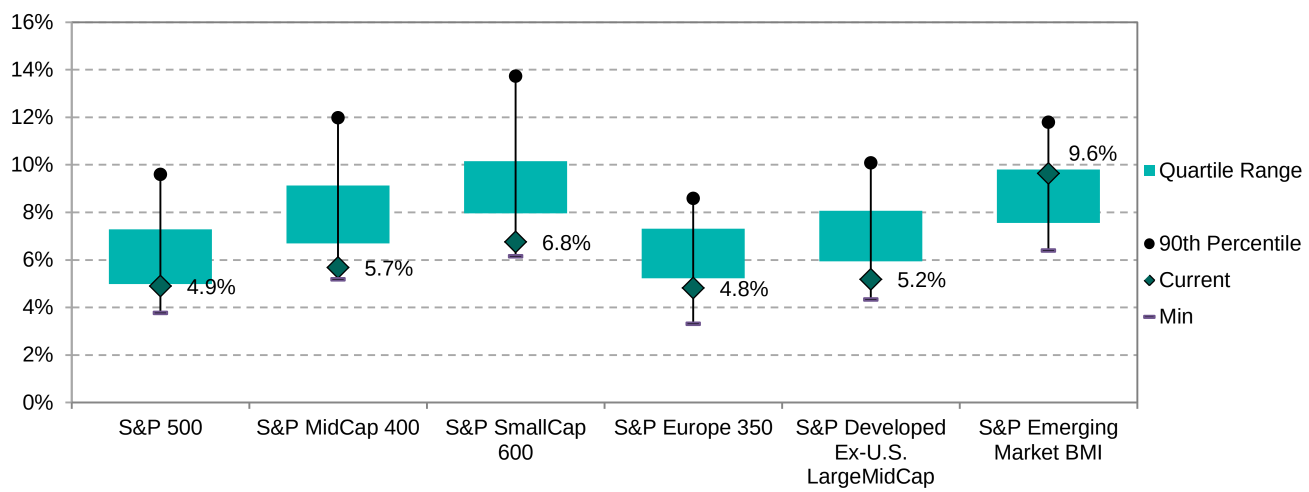
We began by explaining how the benefit of diversification can be measured by the reduction in volatility over a given period, and Equation 1 accounted for this benefit by identifying the idiosyncratic risks (alphas) that are diversified. Equation 2 then justified the use of dispersion to provide a more immediate estimation of this diversification benefit, pointing to how longer-term dynamics might be described via the balance between dispersion and market volatility. Finally in Equation 3 we characterized correlation's role: measuring the relative importance of market volatility versus dispersion in describing overall component volatility. In each case the formulas are approximate, but with a great deal of intuitive coherence and—more importantly—with a useful degree of accuracy in practice.

These characterizations of the three components—volatility, correlation and dispersion—provide **a framework for expectations**: a prediction or expectation for the bounds of any two determines a range for the third. If current levels of dispersion are to increase, either correlation must fall, volatility must rise or an otherwise suitable combination of changes must occur, approximately consistent with the equations above.

The interplay described by these equations connects several key topics of investment concern. Our initial concern was to understand the diversification benefit. But since the diversification benefit is quantified by dispersion, we could equally well have focused on the extent of opportunity available to active managers.[10] These equations produce additional perspectives—for example, correlation as a derived measure of the balance between the strength of idiosyncratic and macro risk factors (see Equation 3). These relationships not only provide insight into the structure of markets, but also describe how they might—and might not—evolve.

Exhibits 5(a) and 5(b) show the recent range of (monthly) dispersions and volatilities registered across several equity markets.

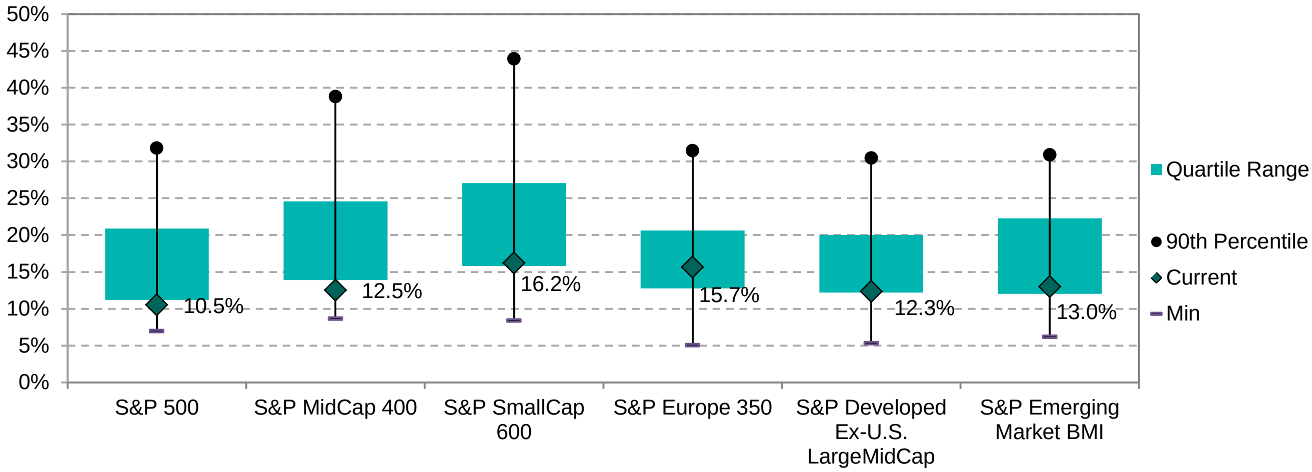
Exhibit 5(a): Index Monthly Dispersion and Historical Context (Jan. 2007 – Mar. 2014)



Source: S&P Dow Jones Indices. Charts are provided for illustrative purposes. Past performance is no guarantee of future results.

[10] See Edwards and Lazzara, *op. cit.*

Exhibit 5(b): Index Monthly Realized Volatility (Annualized) and Historical Context (Jan. 2007 – Mar. 2014)



Source: S&P Dow Jones Indices. Charts are provided for illustrative purposes. Past performance is no guarantee of future results.

It is clear from Exhibit 5(a) that dispersion is currently very low in many developed markets, and returning to our initial theme, the benefit provided by diversification is less than it has been in the past. However, this is not necessarily in itself a particular cause for concern; the current situation has arisen through remarkably low volatility, while correlations have also fallen in each of these markets. Such circumstances have proved resolutely benign, perhaps the "best of all possible worlds:" as this is written, the S&P 500 has broken through several record highs in recent weeks.

Our research leads us to two important conclusions:

1) **If market volatility is to increase, either correlations or individual risk (dispersion) must rise**. If the former occurs, a return to the macro-economic dominance of the previous five years will be a likely culprit. If the latter occurs, the chorus of voices pointing to a bubble in equities will likely gain confidence. (The last such instance accompanied the "tech bubble" of 1999-2002.)

2) **While correlations are indeed currently low, this has not resulted in a large increase in the opportunity set available to active management.** Correlation—as we've remarked previously[11]—is a poor measure of opportunity. The current *combination* of low correlation and low volatility necessarily accompanies low dispersion, from which one can reasonably expect relatively small differences in performance between "good" and "bad" active managers.[12]

These points are not, of course, anticipatory asset allocation advice. We believe that they can guide our expectations for what is *possible* from future market dynamics, narrowing further the boundaries of what we do not know.

[11] Lazzara, Craig, "Dispersion and Correlation: Which is 'Better?'," 30 January 2014,

[12] See Edwards and Lazzara, *op. cit.*, pp 2-3.

APPENDIX A: Proof of Equation 1

Recall that we have the ex-post decomposition of stock returns $R_{i,t} = M_t + \propto_{i,t}$ with market return M_t defined by some fixed weighting system w_i such that $\sum w_i = 1$ and $M_t = \sum w_i R_{i,t}$. The classical CAPM decomposition, allowing for differing market betas, is defined by:

$$R_{i,t} = \beta_i M_t + \epsilon_{i,t} \qquad (1)$$

where the market sensitivity terms β_i are each chosen through a linear regression of market and stock returns. And as a consequence of the regression-based construction, each $\epsilon_{i,t}$ will be statistically independent of the market return; that is:

$$Cov(\epsilon_{i,t}, M_t) = 0 \quad for\ all\ i. \qquad (2)$$

Note that the error terms are related by:

$$\propto_{i,t} = \epsilon_{i,t} + (\beta_i - 1)M_t \qquad (3)$$

Combining (1) with the market portfolio definition and rearranging terms gives:

$$(1 - \sum w_i \beta_i)\, M_t = \sum w_i \epsilon_{i,t} \qquad (4)$$

Applying $Cov(-, M_t)$ to both sides of (4) and utilizing condition (2) gives:

$$Cov\big((1 - \sum w_i \beta_i)\, M_t, M_t\big) = Cov\big(\sum w_i \epsilon_{i,t}, M_t\big) = \sum w_i Cov(\epsilon_{i,t}, M_t) = 0\ .$$

Which implies that $\sum w_i \beta_i = 1$, substitution of which into (4) provides additionally $\sum w_i \epsilon_{i,t} = 0$. In other words, the average β is one and the average ϵ is zero. Now consider the market-weighted average single-stock variance (taken over the full time period):

$$Average\ single\ stock\ variance = \sum w_i Var(R_{i,t}) = \sum w_i Var\left(M_t + \propto_{i,t}\right)$$

Expanding the rightmost term via the additive properties of variance gives:

$$Average\ single\ stock\ variance = \sum w_i Var\,(M_t) + \sum w_i Var\left(\propto_{i,t}\right) + 2\sum w_i Cov\left(M_t, \propto_{i,t}\right)$$

Now, the first term in the expansion is equal to v^2, so it remains only to show that the last term is zero. Via equation (3), the last term can be expanded as:

$$2\sum w_i Cov\left(M_t, \propto_{i,t}\right) = 2\sum w_i Cov\left(M_t, \epsilon_{i,t} + (\beta_i - 1)M_t\right)$$

$$= 2\sum w_i Cov\left(M_t, \epsilon_{i,t}\right) + 2\sum w_i(\beta_i - 1) Cov\,(M_t, M_t) \qquad (5)$$

Recalling that each $Cov\big(M_t,\ \epsilon_{i,t}\big) = 0$ and $\sum w_i \beta_i = 1$, it follows that (5) equates to zero, and therefore:

$$Average\ single\ stock\ variance = Var(M_t) + \sum w_i Var\left(\propto_{i,t}\right)$$

APPENDIX B: Proof of Equation 2

As noted in footnote 9 and evidenced in the exhibits, equality is in general not exact; we prove the special case that each component has an average alpha equal to zero, that is over the sample time periods t = 1, 2, … ,*m*:

$$\sum_{t=1}^{m} \propto_{i,t} = 0 \quad \textit{for all } i$$

This is similar to the condition shown for the ϵ introduced in Appendix A, but without controlling for market betas it is unlikely to be true for a given dataset. The condition is convenient in that it represents many common models for equities where the *expected* average alpha is zero. Then the assumption implies:

$$\frac{1}{m}\sum_{t=1}^{m}\left(\propto_{i,t}\right)^2 = \frac{1}{m}\sum_{t=1}^{m}\left(\propto_{i,t} - 0\right)^2 = Var\left(\propto_{i,t}\right). \qquad (6)$$

By definition of d^2, the average of dispersional variance over time is:

$$d^2 = \sum_{t=1}^{m}\frac{1}{m}\left(\sum_{i=1}^{n} w_i\left(R_{i,t} - M_t\right)^2\right)$$

Since the 1/*m* is a constant, we can bring this inside both summations

$$d^2 = \sum_{t=1}^{m}\left(\sum_{i=1}^{n} w_i\frac{1}{m}\left(R_{i,t} - M_t\right)^2\right)$$

and swap the order of the summation:

$$d^2 = \sum_{i=1}^{n}\left(\sum_{t=1}^{m} w_i\frac{1}{m}\left(R_{i,t} - M_t\right)^2\right).$$

Since each w_i is constant over time, we can take this outside of the inner summation:

$$d^2 = \sum_{i=1}^{n} w_i\left(\frac{1}{m}\sum_{t=1}^{m}\left(R_{i,t} - M_t\right)^2\right)$$

and since by definition $R_{i,t} - M_t = \propto_{i,t}$, this is equal to:

$$d^2 = \sum_{i=1}^{n} w_i\left(\frac{1}{m}\sum_{t=1}^{m} {\propto_{i,t}}^2\right).$$

Finally by (6), we can rewrite the above equation as:

$$d^2 = \sum_{i=1}^{n} w_i Var\left(\propto_{i,t}\right).$$

This expression, combined with Equation 1, gives the intended result.

Appendix C: Proof of Equation 3

Formally, an explicit equality for a tweaked version of Equation 3 holds under more general conditions. The proof may well be older, but can be found in Choueifaty, Froidure and Reynier[13], where the tweaks involve a volatility-weighted version of average correlation and there is an error term that matters in more concentrated portfolios. We give a derivation—limited to a simpler model—that is identical in spirit to an observation made in a much earlier paper by Solnik and Roulet[14], and which captures the essential argument. Note that, opposed to the proofs of Equations 1 and 2, which pertain to *sample data*, the proof here is a statement about *models* involving random variables.

First we model the return $R_{i,t}$ of each asset assets S_i during each time period *t* as the sum of two *random variables*:

$R_{i,t} = M_t + \propto_{i,t}$ (The asset-defining equation)

Where

A. The market return M_t is a random variable with some variance v^2

B. Each $\propto_{i,t}$ is a random variable with mean zero and variance equal to d^2

C. Each $\propto_{i,t}$ is uncorrelated to the market return M_t and for any different pair, $\propto_{i,t}$ is uncorrelated to $\propto_{j,t}$

There are important symmetries to this model that simplify the situation. For example, by symmetry every pair of assets has an equal correlation, equal to some $0 < c < 1$. We can use the same tricks as in the proof of Equation 2 to show that d^2 is both the average *cross-sectional* variance among the $\propto_{i,t}$ and the average *time* variance of the $\propto_{i,t}$.

Proposition: *Let* v^2*,* d^2 *and c be as above; then:*

$$c = \left(\frac{v^2}{v^2+d^2}\right)$$

Proof: Note that condition (C) and the asset-defining equation imply that the return on one asset is uncorrelated to the alpha of another. Without loss of generality, pick a distinct pair of assets S_i and S_j; then:

$$v^2 = Var(M_t) = Cov(M_t, M_t) = Cov(R_{i,t} - \propto_{i,t}, R_{j,t} - \propto_{j,t})$$

Then, the right-hand side of the above decomposes as:

$$v^2 = Cov(R_{i,t}, R_{j,t}) - Cov(R_{i,t}, \propto_{j,t}) - Cov(R_{j,t}, \propto_{i,t}) + Cov(\propto_{i,t}, \propto_{j,t})$$

Note that each of the second and third terms are zero by the introductory remark, and the fourth term is zero by condition (C), so that:

$$v^2 = Cov(R_{i,t}, R_{j,t}). \qquad (*)$$

Now, letting σ indicate standard deviation, the asset-defining equation implies that:

$$\sigma(R_{i,t}) = \sqrt{Var(M_t + \propto_{i,t})}.$$

The additive properties of variance, condition (C), identifying $v^2 = Var(M_t)$ and identifying $d^2 = Var(\propto_{i,t})$ gives:

$$\sigma(R_{i,t}) = \sqrt{v^2 + d^2} \text{ and similarly } \sigma(R_{j,t}) = \sqrt{v^2 + d^2},$$

[13] "Properties of the Most Diversified Portfolio" (2011).
[14] "Dispersion as Cross-Sectional Correlation" (2000).

and since the pairwise correlation *c* multiplied by both standard deviations gives covariance:

$$Cov\left(R_{i,t}, R_{j,t}\right) = c \times (v^2 + d^2).$$

Substituting the above back into the equation (*) gives:

$$v^2 = c \times (v^2 + d^2);$$

rearranging provides the intended result.

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